Hence, if we do not take the similarity transformation ($\alpha_1 = \ldots = \alpha_n = 1$), into account, by the use of known standard procedures we find the following invariants:

$$a_{i1} = \mathrm{Inv},\ i = 1, 2, 3, 4 \tag{3.2}$$

$$\omega_1 [n_3 (q^* - a_{31}q) + n_4 (q^* - a_{41}q)] - \omega_2 [n_1 (q^* - a_{11}q) + n_2 (q^* - a_{21}q)] = \mathrm{Inv}$$

$$(\omega_1 = n_1a_{12} + n_2a_{22},\ \omega_2 = n_3a_{32} + n_4a_{42},\ q^* = n_1a_{11} + \ldots + n_na_{41})$$

Thus, for $h = 1$ and for one resonance relation (1.2), any analytic system of the fourth order can be reduced by a formal transformation to the form

$$x_i = x_i (\lambda_i + a_{i1}u + a_{i2}u^2)$$

where a_{i1} are fixed, while a_{i2} are related by the single condition (3.2).

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ON THE STABILITY OF MOTIONS OF CONSERVATIVE MECHANICAL SYSTEMS UNDER CONTINUALLY-ACTING PERTURBATIONS

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A. Ia. SAVCHENKO
(Donetsk)
(Received July 8, 1973)

We prove some theorems on the stability of motions of conservative mechanical systems under continually-acting perturbations, subject to specified constraints. In the investigation of stability of such type it is usually assumed only that the continually-acting perturbations are small [1]. Such a formulation omits from consideration an important class of conservative systems whose motions do not possess asymptotic stability because an integral invariant exists in them. However, in many problems concerning the structure of the continually-acting perturbations, certain information is available enabling us to estimate their influence

on the stability of motion of a conservative mechanical system [2 – 4]. This question has been discussed in detail in [5 – 8].

1. We are given the system of differential equations

$$dx_s/dt = X_s(t, x_1, \ldots, x_n) \qquad s = 1, \ldots, n \tag{1.1}$$

which allows the particualr solution

$$x_s = 0 \qquad s = 1, \ldots, n \tag{1.2}$$

Concerning the right-hand sides of Eqs. (1.1) we assume that in the region

$$t \geqslant t_1, \qquad x^2 \leqslant H^2 \tag{1.3}$$

they are continuous and allow the existence of a unique solution for specified initial conditions. Here, and everywhere in the following, $x^2 = x_1^2 + \ldots + x_n^2$, $R^2 = R_1^2 + \ldots + R_n^2$. Together with Eqs. (1.1) we consider the system ot equations

$$dx_s/dt = X_s(t, x_1, \ldots, x_n) + R_s(t, x_1, \ldots, x_n), \quad s = 1, \ldots, n \tag{1.4}$$

where the functions R_s characterize the continually-acting perturbations. These functions also are defined and continuous in region (1.3) and satisfy the condition that Eqs. (1.4) have a unique solution under given initial conditions. The following theorem is valid concerning the stability of the solution (1.2) of system (1.1) under continually-acting perturbations R_s.

Theorem 1. The solution (1.2) of system (1.1) is stable under continually-acting perturbations if there exist functions $V(t, x_1, \ldots, x_n)$ and $V_*(t, x_1, \ldots, x_n)$ satisfying the following conditions in region (1.3):

1) V_* is a positive-definite function allowing an infinitely small upper limit;
2) for every $\varepsilon_1 > 0$ there exists $\delta_1(\varepsilon_1) > 0$ such that

$$|V(t, x_1, \ldots, x_n) - V_*(t, x_1, \ldots, x_n)| < \varepsilon_1 \tag{1.5}$$

as soon as $R^2 < \delta_1^2$;

3) for every $\varepsilon_2 > 0$ there exists $\delta_2(\varepsilon_2) > 0$ such that

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} (X_i + R_i) \frac{\partial V}{\partial x_i} \leqslant 0 \tag{1.6}$$

outside the sphere $x^2 < \varepsilon_2^2$ as soon as $R^2 < \delta_2^2$.

Proof. Let $\varepsilon > 0$ be specified. According to condition (1) the inequality

$$V_*(t, x_1, \ldots, x_n) \geqslant W(x_1, \ldots, x_n) \tag{1.7}$$

where W is a positive-definite function not depending explicitly on t, is valid in region (1.3). By α we denote the greatest lower bound of function W on the sphere $x^2 = \varepsilon^2$. Then by virtue of (1.7)

$$V_*(t, x_1, \ldots, x_n) \geqslant \alpha \qquad (t \geqslant t_0) \tag{1.8}$$

everywhere on this sphere. Let $0 < l < \alpha$. In the space of variables $x_1, \ldots, x_n$ we consider the moving surface

$$V_*(t, x_1, \ldots, x_n) = l \tag{1.9}$$

From (1.8) it follows that the inequality $x^2 < \varepsilon^2$ is valid when $t \geqslant t_0$ for all points of this surface. But since the function V_* allows an infinitely small upper limit, for any $t \geqslant t_0$ the surface (1.9) lies between the spheres $x^2 = \varepsilon^2$ and $x^2 = \varepsilon_2^2$, where $0 < \varepsilon_2 < \varepsilon$. We denote this region by Q and its closure by $\bar{Q}$. Condition (3) is fulfilled in region $\bar{Q}$.

Let $x_s = x_s(t)$ $(s = 1, \ldots, n)$ be a solution of system (1.4), satisfying for $t = t_0$ the condition

$$x_s(t_0) = x_{0s} \quad (s = 1, \ldots, n), \quad x_0^2 < \varepsilon_2^2 \tag{1.10}$$

We show that for any $t \geqslant t_0$

$$x^2(t) < \varepsilon^2 \tag{1.11}$$

if $R^2 < \delta^2$, where $\delta > 0$ is some number. We assume the contrary: the trajectory of the solution of system (1.4) with initial conditions (1.10) leaves the ε-sphere $x^2 \leqslant \varepsilon^2$. Then a segment Γ of this trajectory exists starting on the ε_2-sphere and ending on the ε-sphere. Let t_1 and t_2 be the corresponding instants at which the solution trajectory intersects these spheres. Consequently, when $t \in [t_1, t_2]$ the solution trajectory corresponding to the segment Γ belongs wholly to region $\bar{Q}$. Since the moving surface (1.9) belongs to region Q, we have

$$V_*(t_1, x_1(t_1), \ldots, x_n(t_1)) = l_1 < l \tag{1.12}$$

$$V_*(t_2, x_1(t_2), \ldots, x_n(t_2)) = l_2 \geqslant \alpha > l \tag{1.13}$$

Consider the behavior of the time function $V(t, x_1(t), \ldots, x_n(t))$ for $t \in [t_1, t_2]$, where $x_s(t)$ $(s = 1, \ldots, n)$ is a solution of system (1.4) with initial conditions (1.10). Inequality (1.6) is valid in region $\bar{Q}$, therefore, $V(t, x_1(t), \ldots, x_n(t))$ is a nonincreasing function of time on the interval $t \in [t_1, t_2]$. On the other hand, if as ε_1 we take $\varepsilon_1 = (l_2 - l_1)/2$, then the condition (2) for $t = t_1$, with due regard to (1.12), for V we have the upper bound

$$V(t_1, x_1(t_1), \ldots, x_n(t_1)) < (l_1 + l_2)/2 \tag{1.14}$$

while for $t = t_2$, with due regard to (1.13), the lower bound

$$V(t_2, x_1(t_2), \ldots, x_n(t_2)) > (l_1 + l_2)/2 \tag{1.15}$$

Comparing (1.14) and (1.15) we establish that the time function $V(t, x_1(t), \ldots, x_n(t))$ increases along the solution trajectory when $t \in [t_1, t_2]$, which contradicts condition (1.6). Thus, if the solution of system (1.4) satisfies condition (1.10) at $t = t_0$, then the estimate (1.11) is valid for the whole time of the motion if $R^2 < \delta^2 = \min(\delta_1^2, \delta_2^2)$, i.e. solution (1.2) of system (1.1) is stable under such continually-acting perturbations.

Note. In the theorem's hypotheses, besides the usual requirement of smallness of the continually-acting perturbations we have the additional constraint reflected in inequality (1.6).

Corollary. If inequality (1.6) is fulfilled in the whole region (1.3), then the theorem remains also valid when the infinitely small upper limit is absent in the function $V_*(t, x_1, \ldots, x_n)$.

2. We introduce certain definitions. Let system (1.1) have the k integrals

$$V_i(t, x_1, \ldots, x_n) = c_i, \quad i = 1, \ldots, k \tag{2.1}$$

where the V_i are single-valued differentiable functions; moreover, $V_i(t, 0, \ldots, 0) \equiv 0$ for $t \geqslant t_0$, and let system (1.4) have the m integrals

$$V_j'(t, x_1, \ldots, x_n) = c_j', \qquad j = 1, \ldots, m \tag{2.2}$$

with analogous properties.

Definition 1. The continually-acting perturbations R_s preserve the integrals $V_s(t, x_1, \ldots, x_n) = c_s$ $(1 \leqslant s \leqslant k)$ if for every $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $V_j'(t, x_1, \ldots, x_n)$ $(1 \leqslant j \leqslant m)$ such that in region (1.3)

$$| V_s(t, x_1, \ldots, x_n) - V_j'(t, x_1, \ldots, x_n) | < \varepsilon$$

as soon as $R^2 < \delta^2$.

Definition 2. The integral $V_s(t, x_1, \ldots, x_n) = c_s$ $(1 \leqslant s \leqslant k)$ is stable relative to the continually-acting perturbations if for every $\varepsilon > 0$ there exist $\delta_1(\varepsilon, t_0)$, $\delta_2(\varepsilon, t_0) > 0$ such that as soon as

$$R^2 < \delta_1^2, \quad | V_s(t_0, x_1(t_0), \ldots, x_n(t_0)) | < \delta_2$$

it follows that $| V_s(t, x_1(t), \ldots, x_n(t)) | < \varepsilon$. Here $x_i = x_i(t)$ $(i = 1, \ldots, n)$ is a solution of system (1.4).

It is obvious that if the continually-acting perturbations preserve the integral $V_s = c_s$, then it is stable relative to such continually-acting perturbations. Let us assume that the first $p \leqslant m$ integrals (2.1) are preserved. Then, if as the functions V and V_* we take $V = V_1'^2 + \ldots + V_p'^2$ and $V_* = V_1^2 + \ldots + V_p^2$, then they satisfy all the conditions of the corollary to Theorem 1 and, consequently, the conditions for the sign-definiteness of the function V_* are sufficient conditions for the stability of solution (1.2) of system (1.1) under continually-acting perturbations. Thus, in the case under consideration there appears the possibility of using p preserved integrals of system (1.1) to evaluate the stability of solution (1.2).

Pozharitskii [9] has established that the conditions for the sign-definiteness of the function $V_* = V_1^2 + \ldots + V_p^2$ are necessary and sufficient for the existence of some sign-definite function $\varphi(V_1, \ldots, V_p)$ of the known p integrals of system (1.1). Hence follows, in particular, the assertion: if the Liapunov stability of solution (1.2) has been established by constructing a sign-definite function $\varphi(V_1, \ldots, V_k)$ (for example, by the Chetaev method [10]), then this solution is stable under continually-acting perturbations preserving all k integrals. A similar assertion proved by Demin [8] for parametric perturbations of specific type and under more stringent constraints on integrals (2.2).

Example 1. The equations of motion of an absolutely rigid body around a fixed point allow a one-parameter family of stationary solutions. To this family of solutions correspond uniform rotations of the body around certain of its axes, matched with the vertical, with a fixed angular velocity. Sufficient stability conditions for such motions were established in [11] by the Chetaev method [10] of constructing a Liapunov function in the form of a bunch of integrals of the equations of perturbed motion

$$V_1 = A(\xi_1^2 + 2p_0\xi_1) + B(\xi_2^2 + 2q_0\xi_2) + C(\xi_3^2 + 2r_0\xi_3) + 2P(x_0\eta_1 + y_0\eta_2 + z_0\eta_3) = \text{const} \tag{2.3}$$

$$V_2 = \eta_1^2 + \eta_2^2 + \eta_3^2 + 2(\alpha\eta_1 + \beta\eta_2 + \gamma\eta_3) = 0 \tag{2.4}$$

$$V_3 = A(p_0\eta_1 + \alpha\xi_1 + \xi_1\eta_1) + B(q_0\eta_2 + \beta\xi_2 + \xi_2\eta_2) + C(r_0\eta_3 + \gamma\xi_3 + \xi_3\eta_3) = \text{const} \tag{2.5}$$

Let us investigate the stability of such uniform rotations under continually-acting perturbations caused by the action of a small constant gyrostatic moment. Equations (12] corresponding to Eqs. (1.4) allow integrals (2.3), (2.4), and the integral

$$V_3' = A\,(p_0\eta_1 + \alpha\xi_1 + \xi_1\eta_1) + B\,(q_0\eta_2 + \beta\xi_2 + \xi_2\eta_2) + C\,(r_0\eta_3 + \gamma\xi_3 + \xi_3\eta_3) + \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3 = \text{const} \qquad (2.6)$$

Comparing integrals (2.5) and (2.6) we conclude that the continually-acting perturbations of the type being considered preserve integral (2.5) in the sense of Definition 1, i.e. for sufficiently small $\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ the difference $|V_3 - V_3'|$ also is small uniformly relative to $\xi_i, \eta_i\ (i = 1, 2, 3)$ in the region

$$\sum_{i=1}^{3} (\xi_i^2 + \eta_i^2) \leqslant H$$

Consequently, all the uniform relations whose stability was proved [11] by constructing a Liapunov function from integrals (2.3) – (2.5), are stable also under the continually-acting perturbations caused by the small constant gyrostatic moment.

3. Let us consider the case when the first p integrals (2.1) are preserved and the next q ones are stable $(p + q \leqslant k)$ relative to continually-acting perturbations $R_i\ (i = 1, \ldots, n)$.

Theorem 2. If the Liapunov stability of solution (1.2) of system (1.1) has been proved by constructing a Liapunov function from the first $p + q$ integrals (2.1), p of which are preserved and q are stable relative to continually-acting perturbations, then this solution is stable under such continually-acting perturbations.

Proof. Let $\varepsilon > 0$ be specified. Since the Liapunov stability of solution (1.2) of system (1.1) has been proved by constructing a Liapunov function from the first $p + q$ integrals (2.1), by virtue of a theorem in [9] the function

$$V(t, x_1, \ldots, x_n) = \sum_{i=1}^{p+q} V_i^2 (t, x_1, \ldots, x_n) \qquad (3.1)$$

is positive definite, i.e. a positive-definite $W(x_1, \ldots, x_n)$ exists such that

$$V(t, x_1, \ldots, x_n) \geqslant W(x_1, \ldots, x_n) \qquad (3.2)$$

We indicate $l > 0$ such that the surface

$$W(x_1, \ldots, x_n) = l \qquad (3.3)$$

lies wholly inside the sphere $x^2 \leqslant \varepsilon^2$. Since V is a continuous function of the V_i $(i = 1, \ldots, p + q)$, by the theorem's hypotheses it is an integral of system (1.1), stable relative to the continually-acting perturbations being considered. This signifies that for each $l > 0$ there exist $\delta(l, t_0) > 0$, $\delta_1(l, t_0) > 0$ such that as soon as

$$R^2 < \delta_1^2, \quad V(t_0, x_1(t_0), \ldots, x_n(t_0)) < \delta \qquad (3.4)$$

then for all $t \geqslant t_0$

$$V(t, x_1(t), \ldots, x_n(t)) < l \qquad (3.5)$$

where $x_i = x_i(t)\ (i = 1, \ldots, n)$ is a solution of system (1.4). Thus, if the initial data satisfy the condition $x^2(t_0) < \eta^2$, where $x^2 = \eta^2$ is a sphere lying wholly inside the surface $V(t_0\, x_1, \ldots, x_n) = \delta$, then when the first of conditions (3.4) is fulfilled, inequality (3.5) is fulfilled during the whole time of motion. But, with due regard to (3.2), this signifies that the solution of system (1.4) cannot leave the region bounded by surface (3.3) and, consequently, the inequality $x^2(t) < \varepsilon^2$ is fulfilled during the whole time of motion. Q.E.D.

Example 2. Let us consider the problem of the permanent rotations of a heavy rigid body around a fixed point. For the Euler gyroscope it has been proved that the uniform rotations around the major and minor principal axes of its inertia ellipsoid are stable. This fact can be established by constructing a Liapunov function from the energy and areal integrals, geometric integral and from the constancy of the modulus of the angular momentum

$$Ap^2 + Bq^2 + Cr^2 = \text{const} \tag{3.6}$$

Arnol'd [2] has proved that integral (3.6) is stable in the sense of definition 2 under continually-acting perturbations preserving the Hamiltonian structure of the system (for example, parametric perturbations caused by small perturbations of the constructive parameters). Consequently, the uniform rotations around the major and minor principal axes of the inertia ellipsoid of the Euler gyroscope are stable under such continually-acting perturbations by virtue of Theorem 2. This assertion agrees with the results in [2].

For the Lagrange gyroscope the necessary and sufficient condition for the stability of the uniform rotations around its dynamic axis of symmetry is the Maievskii criterion which can be established by constructing a Liapunov function from the energy and areal integrals, geometric integral and from the Lagrange integral $r = \text{const}$. In [13] it was proved that the integral $r = \text{const}$ is stable under continually-acting perturbations of the type described above ($z_0 \neq 0$). Thus, by virtue of Theorem 2 Maievskii's criterion is the stability criterion for such uniform rotations under continually-acting perturbations preserving the Hamiltonian structure of the system, i.e. it is universal in a specific sense.

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ON THE INFLUENCE OF STRUCTURE OF FORCES ON THE STABILITY OF MOTION

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V. M. LAKHADANOV
(Minsk)
(Received January 8, 1973)

We investigate the stability of systems as a function of the structure of the forces which may be dissipative, accelerating, gyroscopic, potential and nonconservative [1].

1. Consider the systems

$$x^{\cdot\cdot} + Dx^{\cdot} + Px = 0 \tag{1.1}$$

$$x^{\cdot\cdot} + Dx^{\cdot} + Px = X(x, x^{\cdot}) \tag{1.2}$$

Here and below x is a column matrix with elements $x_1, \ldots, x_n$; $D = D'$, $P = -P' \not\equiv 0$ are constant ($n \times n$)-matrices; $X(x, x^{\cdot})$ is a column-matrix with elements $X_1(x, x^{\cdot}), \ldots, X_n(x, x^{\cdot})$ containing x_i, $x_i^{\cdot}$ in powers not lower than the second, where $X(0, 0) \equiv 0$. The terms $Dx^{\cdot}$ characterize the dissipative and accelerating forces, the terms Px characterize the nonconservative forces, and the terms $X(x, x^{\cdot})$ characterize the nonlinear forces. We follow everywhere the terminology adopted in [1].

About systems (1.1) and (1.2) we know:

1) system (1.1) is not asymptotically stable [2];
2) systems (1.1) and (1.2) are unstable if $D \equiv 0$ [1, 3];
3) systems (1.1) and (1.2) are unstable if $\mathrm{Sp}\, D < 0$ [2];
4) system (1.1) is unstable if D is sign-positive and the determinant $|P| \neq 0$ [3].

In [3] it was asserted that system (1.1) is unstable for an even n and a sign-positive D. However, the proof carried out in [3] is valid only if $|P| \neq 0$ and, moreover, it is valid in this case for an arbitrary constant matrix D.

We consider the characteristic equation (E is the unit matrix)